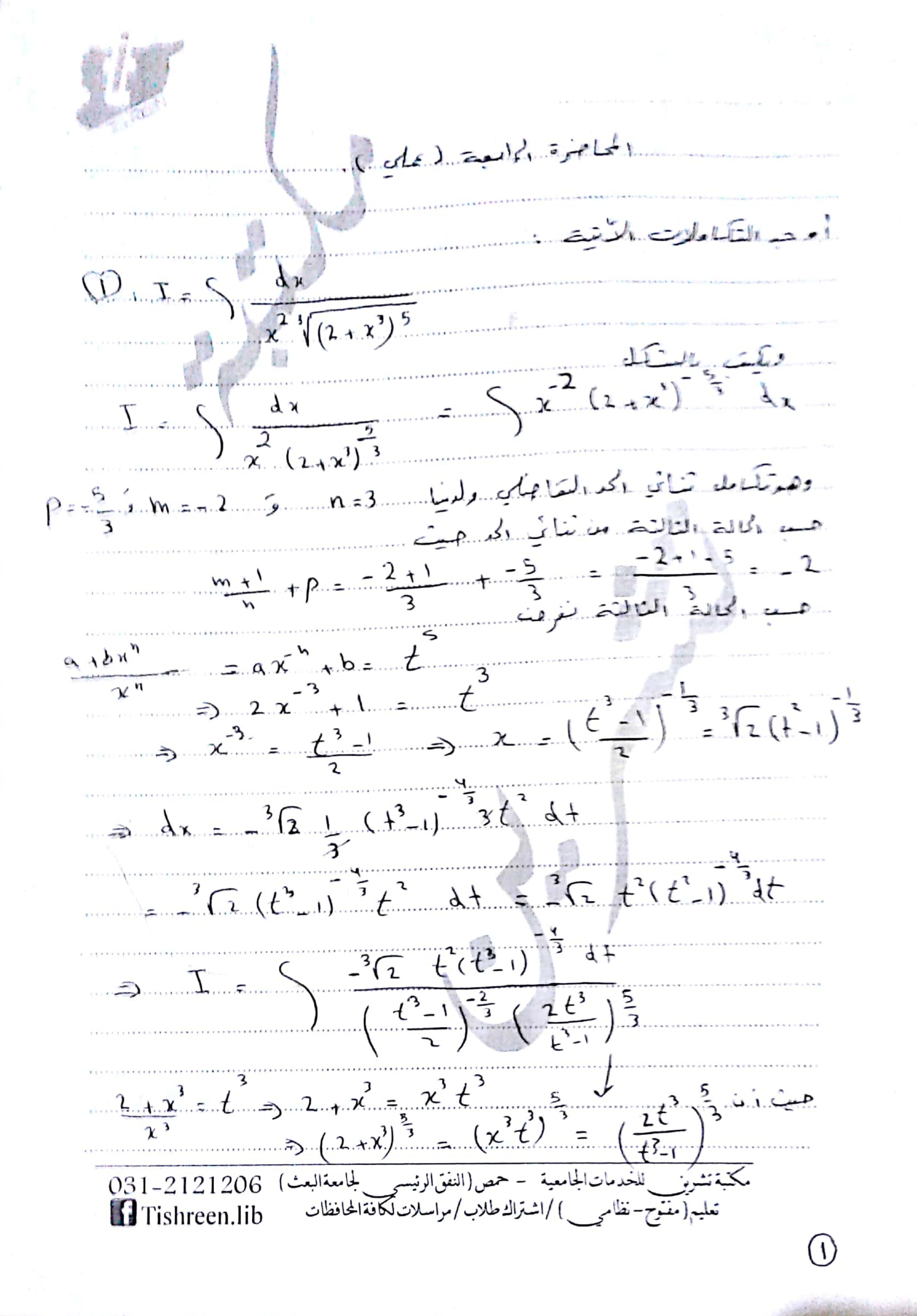


المحاضرة الرابعة (عملي)

أوجد التكاملات الآتية :

① $I = \int \dfrac{dx}{x^2 \sqrt[3]{(2+x^3)^5}}$

ويكتب بالشكل

$I = \int \dfrac{dx}{x^2 (2+x^3)^{\frac{5}{3}}} = \int x^{-2} (2+x^3)^{-\frac{5}{3}} dx$

وهو تكامل ثنائي الحد التفاضلي ولدينا

$p = -\frac{5}{3}$ ، $m = -2$ و $n = 3$

حسب الحالة الثالثة من ثنائي الحد حيث

$\dfrac{m+1}{n} + p = \dfrac{-2+1}{3} + \dfrac{-5}{3} = \dfrac{-2+1-5}{3} = -2$

حسب الحالة الثالثة نفرض

$\dfrac{a+bx^n}{x^n} = a x^{-n} + b = t^s$

$\Rightarrow 2x^{-3} + 1 = t^3$

$\Rightarrow x^{-3} = \dfrac{t^3-1}{2} \Rightarrow x = \left(\dfrac{t^3-1}{2}\right)^{-\frac{1}{3}} = \sqrt[3]{2}\,(t^3-1)^{-\frac{1}{3}}$

$\Rightarrow dx = -\sqrt[3]{2}\,\frac{1}{3}\,(t^3-1)^{-\frac{4}{3}}\, 3t^2\, dt$

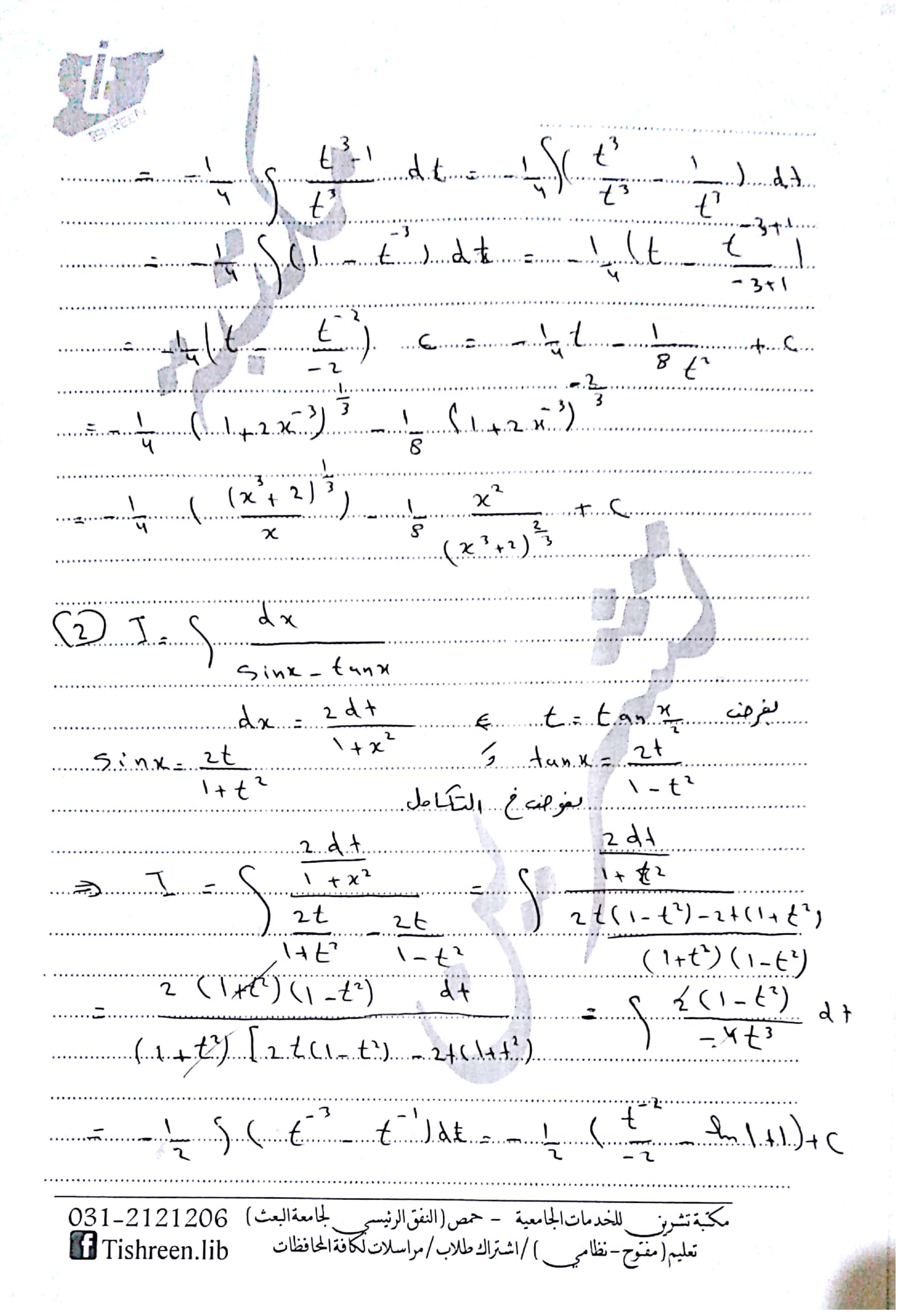
$= -\sqrt[3]{2}\,(t^3-1)^{-\frac{4}{3}}\, t^2\, dt = -\sqrt[3]{2}\; t^2 (t^3-1)^{-\frac{4}{3}} dt$

$\Rightarrow I = \int \dfrac{-\sqrt[3]{2}\;\; t^2 (t^3-1)^{-\frac{4}{3}}\; dt}{\left(\frac{t^3-1}{2}\right)^{-\frac{2}{3}} \left(\frac{2t^3}{t^3-1}\right)^{\frac{5}{3}}}$

حيث أن

$\dfrac{2+x^3}{x^3} = t^3 \Rightarrow 2+x^3 = x^3 t^3$

$\Rightarrow (2+x^3)^{\frac{5}{3}} = (x^3 t^3)^{\frac{5}{3}} = \left(\dfrac{2t^3}{t^3-1}\right)^{\frac{5}{3}}$



$$= -\frac{1}{4}\int \frac{t^3-1}{t^3}\,dt = -\frac{1}{4}\int\left(\frac{t^3}{t^3} - \frac{1}{t^3}\right)dt$$

$$= -\frac{1}{4}\int (1 - t^{-3})\,dt = -\frac{1}{4}\left(t - \frac{t^{-3+1}}{-3+1}\right)$$

$$= -\frac{1}{4}\left(t - \frac{t^{-2}}{-2}\right) + c = -\frac{1}{4}t - \frac{1}{8t^2} + c$$

$$= -\frac{1}{4}(1+2x^{-3})^{\frac{1}{3}} - \frac{1}{8}(1+2x^{-3})^{-\frac{2}{3}}$$

$$= -\frac{1}{4}\left(\frac{(x^3+2)^{\frac{1}{3}}}{x}\right) - \frac{1}{8}\,\frac{x^2}{(x^3+2)^{\frac{2}{3}}} + c$$

(2) $I = \int \frac{dx}{\sin x - \tan x}$

نفرض $t = \tan\frac{x}{2}$ $\in$ $dx = \frac{2\,dt}{1+x^2}$

$\sin x = \frac{2t}{1+t^2}$ و $\tan x = \frac{2t}{1-t^2}$

نعوض في التكامل

$$\Rightarrow I = \int \frac{\frac{2\,dt}{1+x^2}}{\frac{2t}{1+t^2} - \frac{2t}{1-t^2}} = \int \frac{\frac{2\,dt}{1+t^2}}{\frac{2t(1-t^2)-2t(1+t^2)}{(1+t^2)(1-t^2)}}$$

$$= \frac{2(1+t^2)(1-t^2)\,dt}{(1+t^2)\left[2t(1-t^2)-2t(1+t^2)\right]} = \int \frac{2(1-t^2)}{-4t^3}\,dt$$

$$= -\frac{1}{2}\int (t^{-3} - t^{-1})\,dt = -\frac{1}{2}\left(\frac{t^{-2}}{-2} - \ln|t|\right) + c$$

$$= -\frac{1}{2}\frac{1}{\tan^2\frac{x}{2}} - \frac{1}{4}\ln\left|\tan\frac{x}{2}\right| + c$$

(3) $$I = \int \frac{2\sin x + \cos x}{\sin^2 x \cos x + 4\cos^3 x}\,dx$$

نقسم البسط والمقام على $\cos^3 x$

$$I = \int \frac{\frac{2\sin x + \cos x}{\cos x} \cdot \frac{1}{\cos^2 x}}{\frac{\sin^2 x \cos x}{\cos^3 x} + 4\frac{\cos^3 x}{\cos^3 x}}\,dx$$

حيث $R(-\sin x, -\cos x) = R(\sin x, \cos x)$

نفرض $t = \tan x$ ← $dt = \frac{1}{\cos^2 x}\,dx$ نعوض

$$\Rightarrow I = \int \frac{(2\tan x + 1)\cdot dt}{\tan^2 x + 4} = \int \frac{(2t+1)\,dt}{t^2+4}$$

$$= \int \frac{2t}{t^2+4}\,dt + \int \frac{dt}{t^2+4}$$

$$= \ln|t^2+4| + \frac{1}{2}\arctan\frac{t}{2} + c$$

يمكن حل التمرين بفرضية أخرى

وذلك بأن نقسم البسط والمقام على $\cos x$ فيصبح لدينا

$$I = \int \frac{2\tan x + 1}{\sin^2 x + 4\cos^2 x}\,dx$$ نفرض $\tan x = t$

$$\Rightarrow I = \int \frac{2t+1}{\frac{t^2}{1+t^2} + \frac{4}{1+t^2}}\,\frac{dt}{1+t^2} = \int \frac{2t+1}{t^2+4}\,dt$$

$$= \ln|t^2+1| + \frac{1}{2}\arctan\frac{x}{2} + C$$

(4) $$I = \int \frac{\tan x + \tan^3 x}{1+\tan^3 x}\,dx$$

نفرض $\tan x = t$

$\Leftarrow x = \arctan t \Leftarrow dx = \dfrac{dt}{1+t^2}$

$$\Rightarrow I = \int \frac{t+t^3}{1+t^3}\cdot\frac{dt}{1+t^2} = \int \frac{t(1+t^2)}{1+t^3}\cdot\frac{dt}{1+t^2}$$

$$= \int \frac{t}{1+t^3}\,dt$$

درجة البسط أصغر من درجة المقام نفرق الكسر

$$\frac{t}{1+t^3} = \frac{t}{(1+t)(t^2-t+1)} = \frac{A}{1+t} + \frac{(Bt+C)}{(t^2-t+1)}$$

$$\Rightarrow t = A(t^2-t+1) + (Bt+C)(1+t)$$

$$\Rightarrow t = At^2 - At + A + Bt + Bt^2 + C + Ct$$

$$\Rightarrow t = At^2 + Bt^2 - At + Bt + A + C + Ct$$

$A + B = 0 \Rightarrow A = -B$ و $A + C = 0 \Leftarrow A = -C$

و $-A + B + C = 1$

$\Rightarrow -A - A - A = 1 \Rightarrow -3A = 1$

$\Rightarrow A = -\dfrac{1}{3}$

وبالتالي $-B = A \Leftarrow B = -A = \dfrac{1}{3}$

و $C = \dfrac{1}{3}$

$$I = \int \frac{-\frac{1}{3}}{1+t} + \frac{(\frac{1}{3}t + \frac{1}{3})}{t^2-t+1}\,dx$$

$$= \frac{1}{3}\ln|x+1| + \frac{1}{2}\ln|x^2-x+1| + c$$

$$+ \frac{1}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} + C$$

(5) $I = \int \sin^4 x \cos^4 x \, dx$

$$= \int \left(\frac{\sin 2x}{2}\right)^4 dx = \frac{1}{16}\int \sin^4 2x \, dx$$

$$= \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 dx$$

$$= \frac{1}{64}\int \left(1 - 2\cos 4x + \cos^2 4x\right) dx$$

$$= \frac{1}{64}\int \left(1 - 2\cos 4x + \frac{1+\cos 8x}{2}\right) dx$$

$$= \frac{1}{64}\left[\frac{3}{2}x - \frac{1}{2}\sin 4x + \frac{1}{16}\sin 8x\right] + C$$

«انتهت المحاضرة الرابعة»

«مع تمنياتي لكم بالتوفيق والنجاح»

إعداد: [illegible]